Essay

by
John H. Hubbard
Department of Mathematics, Cornell University

CHAOS

A famous science fiction story tells of a politician who, soon after winning an election, takes a trip in a time travel machine back to the days of the dinosaurs. While there, and though much admonished not to disturb anything, he bends a blade of grass. When he returns to his real present, he finds that in this modified world he has lost the election.

This is what mathematicians mean when they say that a system exhibits chaos: Tiny changes in the initial state of a system can decisively affect the outcome. One speaks of the butterfly effect. Can the fluttering of the wings of a butterfly in Japan have a decisive effect on the weather in the United States a month later?

Most people would probably dismiss the suggestion as preposterous without a second thought. But I think it is true, at least if the time span of one month is increased to six months, and I propose here to give some quantitative reasons for my conclusions.

It is not obvious how you would go about justifying the reality of the butterfly effect. We do not have a time machine available; we cannot go back six weeks, catch a butterfly (while disturbing nothing else, whatever that means), and then come back and observe the consequences. We need to take a more devious tack.

To help you follow the argument, I will describe a "toy model" that clearly exhibits "butterfly behavior," and within which the notions that I will introduce can be understood.

Consider the (purely mathematical) system in which, at each tick of the clock, an angle is doubled. A state of the system is an angle, and it evolves by doubling again and again. In symbols, you could describe the system as a sequence of angles,

$$\theta_0, \theta_1, \dots,$$

where θ_0 is the initial state of the system and $\theta_{n+1} = 2\theta_n$.

This system exhibits butterfly behavior. If θ_0 is perturbed by one-billionth of a turn, then the state after 30 ticks of the clock is completely unknown. In-

deed, the uncertainty in our knowledge of the state of the system doubles at each tick; after 30 ticks, our uncertainty has grown to

$$\frac{2^{30}}{1{,}000{,}000{,}000},$$

which is slightly greater than 1. Our uncertainty is now more than a turn; we don't know anything anymore.

The example above brings in a key notion in all descriptions of chaos entropy. This is essentially the rate at which information dissipates. There are many ways of describing this rate with precision, and they go by various names (for example, Lyapunov exponents), but for the purposes of this article, I will stick with the doubling time: **the amount of time it takes for a small uncertainty to double.**

How would we estimate this time for the system formed by the weather? Of course, we cannot "choose two initial states for the weather an epsilon apart and measure how fast they diverge," but we can do something like it. We can go back over history and find times when the weather was very similar. Then we can see how long it took for the weather pattern to diverge. This has been done and leads to a doubling time of about $2\frac{1}{2}$ days.

You can also go to the meteorology department of a big research institution (in practice, the Institut de Météorologie de l'université de Paris–VI) and ask what the doubling time is for their best computer models. You get much the same figure.

The next question that must be faced is this: What corresponds to the number one-billionth above? What proportion of the size of the system (the atmosphere) is our disturbance (one butterfly)? One (perhaps contestable) way of estimating this is simply to measure the ratio of the masses. We might guess that a butterfly weighs 1 gram (rather a heavy butterfly), and it turns out that the atmosphere has a mass of about 5×10^{21} grams.

The atmospheric pressure is very nearly 1 kg/cm^2, meaning that there is 1 kg of air above every square centimeter of the earth. The area of a sphere of radius r is $4\pi r^2$, and the radius of the earth is about 6000 km. Thus the mass of the atmosphere is about

$$1000 \times 4 \times \pi \times (6 \times 10^8)^2 \text{ grams.}$$

So a butterfly isn't one-billionth the size of the system; it is more nearly a thousand-billion-billionth.

Since 5×10^{21} is approximately 2^{72}, it should take about 72 doubling periods for the effects of an individual butterfly to induce perturbations on a global scale.

One consequence of this analysis is that long-term weather prediction should be completely impossible. It is inconceivable that anyone could ever know the state of the atmosphere to within the effect of one butterfly, or even on a scale a thousand billion times larger. Perturbations on that scale decisively affect the global weather within a month.

Physicists, chemists, astronomers, and mathematicians have now shown that a vast collection of systems appear to display "chaos," in the sense that they are expanding and have a doubling time for errors.

One example is due to the meteorologist E. Lorenz, whose discovery may be credited with starting this entire line of thought. In 1961 he was running a simulation of the weather. The computers of the time were primitive, so the data had to be drastically simplified for any computation to be possible. He observed various behaviors from his model, apparently quite satisfactory, until one day he decided to examine something he had already computed over a longer time. He typed in what he thought were the original initial conditions, went off to get a cup of coffee, and when he came back he found that his new "weather" was unrelated to the previous run of his model.

He eventually realized that the difference was due to the fact that he had entered the initial conditions with fewer decimals than in the initial run. After further simplification of his model, Lorenz found that the following set of three coupled differential equations exhibits the same sort of chaotic behavior:

$$x' = 10(y - x)$$
$$y' = 28x - y - xz$$
$$z' = \frac{8}{3}z + xy.$$

Much further work on these equations and others in R^3 have shown that chaotic behavior and fractal attractors are common.

The presence of chaos plays havoc with predictions, but it is sometimes useful; sometimes chaos can be translated into control.

NASA is not able to build rockets with enough fuel to give the speeds required to go huge distances. So they delicately bounced the spacecraft off Venus, stealing a bit of Venus's potential energy to give the spacecraft the fantastic speeds required. Only because the tiny variations in the trajectory due to the guidance rockets can be amplified to enormous variations in speed and trajectory is such a scheme feasible. But imagine how difficult this makes the long-term predictions about the orbits of comets.

The presence of chaos also has philosophical consequences, for instance, about the conflict of determinism and free will. How can humans have free will if the universe is completely governed by deterministic laws?

Well, if the equations exhibit chaos, then it follows that you cannot know that they are deterministic, however long you observe the system. If you were to observe a sequence of "doubled angles," each with 15 decimal places, you could never know whether you are observing a sequence of exact doubles of some angle or the same system perturbed at a scale smaller than 10^{-15} (such as round-off error).

Similarly, if the brain were not following exactly the laws of physics but were perturbed (by spiritual forces, free will, god) at a scale unmeasurable without drastically affecting the system (I doubt anyone reacts the same way with electrodes in his brain), then you could never know it, but over a time scale of perhaps 4 seconds, you could decisively alter all decisions. Making precise sense of the above would require knowing something about the doubling time of the brain. Introspection tells me this should be perhaps 0.1 second, the time of an elementary realization. Perhaps some day neurology will give a more credible estimate. If this is roughly accurate, then 4 seconds is 40 doubling times later, and $2^{40} \approx 10^{12}$ is approximately the number of neurons in the brain.

Personally, I do not think that there is a god tinkering with the laws of physics in my brain, but this is not knowable. Chaos prevents us from knowing such things.

More generally, although I am as firm a scientist as you could hope to find, I think that the world is essentially incomprehensible, with all sorts of tiny events having enormous consequences that no one can hope to foresee or understand.

If you find preposterous the idea that your smallest actions probably influence the entire world after a short time, consider the following. The time span used is much longer, about 1200 years, but I think the implications are much the same. Who were your ancestors in the year 800 A.D.? You have two parents, four grandparents, and so on. The number of ancestors should be 2^n in the nth generation. The year 800 is 50–60 generations ago, leading to $2^{50} \approx 10^{15}$ ancestors at that time. Of course this is a million times larger than the population of the world today, and a much larger multiple of the population at that time. This shows among other things that there must have been a lot of inbreeding.

Now think about it from the point of view of a woman in the year 800. In order for the population to be sustained, and even grow, she had to have at least an average of 2 children, 4 grandchildren, and so on. Of course, her line might become extinct, but if that were to happen, then it must happen soon, after fewer than three generations. The probability of having no descendants can be evaluated over the short term and might be $\frac{1}{3}$ or $\frac{1}{2}$. And once the descendance has become large, it tends to become the entire population quite rapidly (although the influence of inbreeding slows this down considerably).

If you put these two points of view together, you will see that essentially everyone of European ancestry is descended from every person alive in Europe in the year 800 who has any descendants at all. For instance, we are all descended from Charlemagne, but also from about half of the peasants alive at the time. If just one of those couples had had one child fewer, *everyone* would be genetically different.

Of course, this doesn't make sense; long ago the paths of history would have diverged so that no detailed comparisons are possible. But certainly the smallest variation in the sexual behavior of anyone at that time would certainly have affected the world in incalculable ways.